

# Anisotropic brane cosmologies with exponential potentials

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We study Bianchi I type brane cosmologies with scalar matter self-interacting through combinations of exponential potentials. Such models correspond in some cases to inflationary universes. We discuss in detail the conditions for accelerated expansion to occur: in particular, we show that the condition which is necessary and sufficient for inflation in the relativistic version of the models is not sufficient in the brane case. Another peculiar feature of the models is that the relationship between the value of the scale factor at the beginning of inflation and the equation of state is very different from what one finds in the relativistic framework. We also analyze the influence of the value of the anisotropy and the brane tension, and show that the associated effects become negligible in the late time limit, those related to the anisotropy disappearing earlier. This study focuses mainly on single field models, but we also consider a generalization yielding models with multiple non-interacting fields and examine their features briefly. We conclude that, in the brane scenario, an increase in the number of fields assists inflation, as happens in general relativity.

## I. INTRODUCTION

According to the most promising candidates for a quantum gravity theory, we are living in more than four dimensions [1], and gravity is effectively four dimensional only at low enough energies. This has inspired scenarios in cosmology different from the standard one. The “brane world” picture is the most recent of them all [2], and assumes that ordinary matter is confined to a hypersurface with three spatial dimensions embedded in a multidimensional space-time, allowing gravity act in the fifth dimension.

The “brane-world” proposal has motivated a revision of the predictions to be tested by future observations. Even if in a first approach isotropy is assumed, one will immediately find [3] that the evolution of the Universe in this new scenario is rather peculiar, at least at early times, because of the modifications arising in the Hubble equation. Given this, one may wonder for instance what aspects of cosmological inflation should be changed to fit in this alternative description. Clearly, one of the most important problems to address within this framework is that of the initial conditions. Recently, it has been suggested [4] that anisotropic spacetimes typically expand faster than isotropic ones. This study was grounded on a qualitative analysis of the evolution equations for the scalar field (Klein-Gordon equation) and the average expansion rate (generalized Raychaudhuri equation) for Bianchi I type cosmologies.

In this spirit, we have investigated the same equation set, together with its solutions, just assuming at the beginning that the potential depends on the scale factor in such a way that it is compatible with inflation. In particular, we have found that, under some assumptions, the matter source can be cast in the form of a scalar field self-interacting through a combination of exponential potentials. Such models were first considered in the

context of inflation nearly two decades ago [5], and have been paid much attention along the years because of several reasons. First of all, there is solid motivation that makes them physically appealing, for they appear in four-dimensional effective Kaluza-Klein theories arising from compactification of higher dimensional supergravity or superstring theories (see [6] for a review). In addition, it was shown long ago that FRW power law models, which are driven by exponential potentials, behave as attractors in the sense that models with broad disparity in their initial conditions become power law solutions at late times [7, 8]. Last but not least, exponential potentials have proved very useful in general relativity for providing new exact solutions because they add a rather small degree of non-linearity to the field equations [9]. Unfortunately, the equations one has to solve in brane cosmology are intrinsically so non-linear that, even if one considers just exponential potentials, the task of finding exact solutions is a hard one, except in very specific cases.

With the aim of getting round this difficulty, and as mentioned before, we will not assume from the start that we have an exponential potential or a combination of such terms, but rather end up having it after making a certain ansatz regarding time evolution.

As expected, if the expansion is fast enough then anisotropy exerts less influence on the expansion than the quadratic energy-momentum corrections due to the extrinsic curvature. It follows then that anisotropy does not prevent inflation, as one also gets in the framework of general relativity [10, 11, 12]. In the late time regime, the effect of anisotropy and extra dimensions are negligible, and the solutions behave as if they were isotropic and purely relativistic.

The qualitative behavior of Bianchi I type brane cosmologies has been studied using dynamical systems techniques by several authors. Barotropic perfect fluids were considered by Campos and Sopuerta in the absence [13]

and presence [14] of bulk effects; whereas Goheer and Dunsby [15] considered models with a single scalar field (see also [16] for an earlier study along the same lines focusing on isotropic cosmologies). According to those studies, late time isotropization is an inherent feature to the models. This idea of inflation acting as an isotropizing agent, that was put forward years ago by Wald [17], has been recently revisited in the brane scenario [18] (see also [19, 20] for a discussion on the isotropization of anisotropic relativistic cosmologies with an exponential potential). Nevertheless, we have recently shown [21] that the brane might interact with the bulk in a way not compatible with late time isotropization. Here, however, we will discard the bulk effects and Wald's argument will still apply.

In another recent study [22], Bianchi I type exact solutions in parametric form were found for barotropic perfect fluids and several scalar matter models. It was shown that for exponential potentials the simple relation  $\phi \propto \log t$  will not hold at all times. This does not exclude, though, the possibility of having this behavior at some epoch, and that is indeed the case for the late time regime of our solutions. We also show here that the effects of the extra dimensions make the models isotropize faster, this being linked to an increase in the Hubble rate.

Another interesting aspect of the problem is that, typically, exponential potentials only render inflation if their slope is small enough. Since our solutions mimic their relativistic counterparts late in their history, it is not surprising that the same restrictions on the values of the slope will hold for our models too. This is discussed in Section 3. However, one could still have inflation with sloppy potentials if there are enough non-interacting fields. This is the so called assisted inflation proposal. We also analyze this possibility in the last section, and find that inflation is likelier in such multifield models because the Hubble factor grows as the number of fields increases.

The plan of the paper is as follows: in Sec. 2 we study the equations of motion for the scalar field and the gravitational field equations discarding bulk effects and under the assumption of a Bianchi type I geometry. In Sec. 3 we consider solutions such that their potential energy decreases as the models expand, and find approximate solutions with power law inflation at very late times. Sec. 4 is devoted to discussing in detail the inflationary behavior, paying special attention to the role of anisotropy and the corrections from extra dimensions. In this section, we show that the value of the expansion factor at the beginning of inflation depends on the equation of state in an peculiar way, if we compare the results with what one gets in the relativistic. This suggests that the evolution of brane cosmologies is in some aspects very different from what is regarded as standard, and that the subject deserves further investigation. In Sec. 5 we turn to consider multi-field generalizations and examine their features. We show that it is possible to model assisted inflation in the brane setup too. Finally, in Sec. 6 we

draw our main conclusions, and review future prospects.

## II. GEOMETRY ON THE BRANE AND FIELD EQUATIONS

We assume that the 5-dimensional line element giving the geometry of the bulk is of the form [23]

$$d\tilde{s}^2 = \tilde{g}_{AB} dx^A dx^B, \quad (1)$$

where  $A, B = 0, 1, 2, 3, 4$ . Tildes will be used to denote the bulk counterparts of standard general relativistic quantities. If we now denote by  $n^A$  the unit vector normal to the brane, then the metric induced on that hypersurface is  $g_{AB} = \tilde{g}_{AB} - n_A n_B$ . It is convenient to define a new coordinate  $\chi$  such that the brane is located at  $\chi = 0$  and  $n_A dx^A = d\chi$ . This renders the line element in the alternative form

$$ds^2 = d\chi^2 + g_{\mu\nu} dx^\mu dx^\nu \quad (2)$$

with  $\mu, \nu = 0, 1, 2, 3$ . The five-dimensional bulk field equations are

$$\tilde{G}_{AB} = -\tilde{\kappa}^2 [\tilde{\Lambda} \tilde{g}_{AB} + \delta(\chi)(\lambda g_{AB} - T_{AB})]. \quad (3)$$

Effectively, the Einstein equations on the brane take the form

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - E_{\mu\nu}, \quad (4)$$

where

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu{}^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2), \quad (5)$$

$\Lambda = \tilde{\kappa}^2 (\tilde{\Lambda} + \tilde{\kappa}^2 \lambda^2 / 6) / 2$ ,  $\kappa^2 = \tilde{\kappa}^4 \lambda / 6$ ,  $T = T_\alpha^\alpha$ , and  $E_{AB} = C_{AIBJ} n^I n^J$ . Here  $C_{AIBJ}$  is the five dimensional Weyl tensor in the bulk, whereas  $T_{\mu\nu}$  and  $\lambda$  stand for the values on the brane of the energy-momentum tensor and the vacuum energy respectively. The equations were obtained under the assumption of  $\tilde{Z}_2$  symmetry, and confinement of the matter fields to the brane, in agreement with the brane scenario devised by Horava and Witten [24].

It has been shown that  $E_{\mu\nu}$  can be decomposed like

$$E_{\mu\nu} = -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \left[ \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + 2 \mathcal{Q}_{(\mu} u_{\nu)} \right], \quad (6)$$

where  $u^\mu$  is the four velocity of an observer on the brane,  $\mathcal{U}$  is the effective non local energy density on the brane,  $\mathcal{P}_{\mu\nu}$  is the effective nonlocal anisotropic stress, and  $\mathcal{Q}_\mu$  is an effective non local energy flux on the brane, which vanishes identically for a Bianchi type I brane. In addition  $\mathcal{P}_{\mu\nu}$  and  $E_{\mu\nu}$  are subject to the constraints

$$D^\nu \mathcal{P}_{\mu\nu} = 0, \quad (7)$$

$$\nabla^\mu E_{\mu\nu} = \tilde{\kappa}^4 \nabla^\mu S_{\mu\nu}. \quad (8)$$

Let us consider now four-dimensional Bianchi I type spacetimes. The line element is given by

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2. \quad (9)$$

As usual, we define the expansion rates along the three spatial directions as  $H_i = \dot{a}_i/a_i$  for  $i = 1, 2, 3$ . The shear tensor will be denoted as  $\sigma_{\mu\nu}$  and

$$\sigma^{\mu\nu}\sigma_{\mu\nu} = \sum_{i=1}^3 (H_i - H)^2, \quad (10)$$

where  $3H = H_1 + H_2 + H_3$ . For this metric, Eq. (7) is identically satisfied. Moreover, since there is no evolution equation for  $\mathcal{P}_{\mu\nu}$  on the brane, one usually assumes that it is either null or satisfies the less restrictive condition that  $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$  (It has been proved [25] that the integrability conditions for  $\mathcal{Q}_\mu = 0$ ,  $\mathcal{P}_{\mu\nu} = 0$  imply spatial homogeneity, so this is consistent with having a Bianchi I metric on the brane.). Both assumptions on  $\mathcal{P}_{\mu\nu}$  transform Eq. (8) into a very simple evolution equation for  $\mathcal{U}$ , namely

$$\dot{\mathcal{U}} + 4H\mathcal{U} = 0. \quad (11)$$

The solution of the latter corresponds to non local energy evolving like radiation, i.e.  $\mathcal{U} = \mathcal{U}_0/a^4$ , with  $\mathcal{U}_0$  a constant that can be null. As we see, the joint choice of  $\mathcal{P}_{\mu\nu} = 0$  and  $\mathcal{U} = 0$  is consistent on the brane. Whether this is also fully consistent for the bulk remains an open question, but we follow other authors in making these arguably strong assumptions [4, 26]. This allows gaining considerable insight in the changes on the evolution of cosmological models associated with the modification of the Friedmann equation, which might be otherwise concealed by bulk effects.

Under the assumptions made, the gravitational field equations and the Klein-Gordon equation read

$$H_1 H_2 + H_1 H_3 + H_2 H_3 = \Lambda + \kappa^2 \rho + \frac{\tilde{\kappa}^4}{12} \rho^2, \quad (12)$$

$$H_2^2 + H_2 H_3 + H_3^2 + \dot{H}_2 + \dot{H}_3 = \Lambda - \kappa^2 p - \frac{\rho}{12} (2p + \rho) \tilde{\kappa}^4, \quad (13)$$

$$H_1^2 + H_1 H_3 + H_3^2 + \dot{H}_1 + \dot{H}_3 = \Lambda - \kappa^2 p - \frac{\rho}{12} (2p + \rho) \tilde{\kappa}^4, \quad (14)$$

$$H_1^2 + H_1 H_2 + H_2^2 + \dot{H}_1 + \dot{H}_2 = \Lambda - \kappa^2 p - \frac{\rho}{12} (2p + \rho) \tilde{\kappa}^4, \quad (15)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \quad (16)$$

where we have set

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (17)$$

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (18)$$

according to the customary interpretation of irrotational scalar fields in terms of a perfect fluid.

Due to the isotropy of the matter source, one can obtain constraint equations which provide a relationship expressing the expansion rate along one direction in terms of the other two and their derivatives.

Even though the solutions to the system (12)–(15) include locally rotationally symmetric models (LRS), i.e., models with the same expansion rate along two directions, we will not consider them any further. Thus, in what follows we will assume  $H_1 \neq H_2$ ,  $H_2 \neq H_3$ ,  $H_3 \neq H_1$ , so that the models admit only three isometries. The constraints arising from having identical pressure along the three spatial directions are

$$H_1 + H_2 + H_3 = -\frac{\dot{H}_2 - \dot{H}_1}{H_2 - H_1} = -\frac{\dot{H}_3 - \dot{H}_2}{H_3 - H_2} = -\frac{\dot{H}_1 - \dot{H}_3}{H_1 - H_3}. \quad (19)$$

This allows to distinguish two different cases. The first one corresponds to  $\dot{H}_2 \neq \dot{H}_3$ . Hence  $\dot{H}_1 \neq \dot{H}_2$  and  $\dot{H}_3 \neq \dot{H}_1$ . Integration of Eq. (19) yields the following relationships:

$$H_1 - H_3 = \vartheta(H_1 - H_2) = \frac{\vartheta}{\vartheta - 1}(H_2 - H_3), \quad (20)$$

where  $\vartheta \neq 1$  is an arbitrary integration constant. The second case arises when the Hubble factors along any two directions differ only by a constant, then necessarily  $H_1 + H_2 + H_3 = 0$  and we get a model which does not evolve on average. For simplicity we will discard these cases too.

It is convenient to write the field equations using the following definitions:

$$P \equiv H_1 - H_2, \quad (21)$$

$$Q \equiv H_1 - H_3. \quad (22)$$

We will also use the following constant:

$$\varpi \equiv \frac{1 - \vartheta + \vartheta^2}{3} \geq \frac{1}{4}. \quad (23)$$

Note that, since we are not considering LRS models, we will have  $\varpi \neq 1/3$ . Equations (12)–(16) become

$$Q = \vartheta P, \quad (24)$$

$$H = -\frac{\dot{P}}{3P}, \quad (25)$$

$$\Lambda + \kappa^2 \rho + \frac{\tilde{\kappa}^4}{12} \rho^2 = 3H^2 - \varpi P^2, \quad (26)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \quad (27)$$

### III. COSMOLOGIES WITH EXPONENTIAL POTENTIALS

In this section we look for solutions by means of an unconventional technique that consists in taking the average scale factor  $a = (a_1 a_2 a_3)^{1/3}$  as the independent

variable, instead of the time variable  $t$ . This procedure has been successfully used in related problems.

Taking advantage of the insight gained in Ref. [27], we write

$$V(\phi(a)) = \frac{F(a)}{a^6} \quad (28)$$

and make the change of variables  $dt = a^3 d\eta$  in Eq. (27), so that

$$\frac{d^2\phi}{d\eta^2} + a^6 \frac{dV}{d\phi} = 0, \quad (29)$$

which in turn gives us the first integral

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{6}{a^6} \int \frac{F}{a} da = \frac{c}{a^6}, \quad (30)$$

where  $c$  is an arbitrary integration constant. This is equivalent to saying that

$$\rho(a) = \frac{6}{a^6} \int \frac{F}{a} da + \frac{c}{a^6}. \quad (31)$$

Combining  $H = \dot{a}/a$  with Eq. (25) we get  $P = c'a^{-3}$ , where  $c'$  is another arbitrary integration constant that we will set to one for simplicity. Now, using Eqs. (25) and (26) we can formally reduce the problem to quadratures, namely

$$t = t_0 \pm \int \frac{\sqrt{3} da}{a \sqrt{\Lambda + \kappa^2 \rho(a) + \frac{\tilde{\kappa}^4}{12} \rho(a)^2 + \frac{\varpi}{a^6}}} \quad (32)$$

and

$$\phi = \phi_0 \pm \int \frac{\sqrt{6}}{a} \frac{\sqrt{\rho(a) - \frac{F}{a^6}} da}{\sqrt{\Lambda + \kappa^2 \rho(a) + \frac{\tilde{\kappa}^4}{12} \rho(a)^2 + \frac{\varpi}{a^6}}}, \quad (33)$$

where  $t_0, \phi_0$  are integration constants. We will choose the plus sign in Eq. (32) so that we have expansion as time grows, and will also make the same sign choice in Eq. (33) for reasons that will become clear in the lines below.

Our main objective at this stage is to find expressions for  $F(a)$  so that they are suitable for depicting a given kind of inflationary behavior, and in particular that associated with exponential potentials. We assume then that the expansion is driven by a scalar field striving to make its potential energy as small as possible. If the potentials have positive slopes, the only possibility is that the scalar field grows with time, and in particular with the scale factor. Therefore, we conclude that given our purposes we have to choose the plus sign in Eq. (33).

One simple way of satisfying the requirements we just made on the potential is to set

$$F(a) = f a^n \quad (34)$$

with  $n < 6$  and  $f > 0$ . This gives

$$V(a) = f a^{n-6}. \quad (35)$$

Note that, as pointed out in Ref. [23], the consequence of the usual assumption  $\tilde{\Lambda} < 0$  allows for an arbitrary value of  $\Lambda$ , and in particular it can be chosen so that it vanishes or takes some other wanted value. Let us set  $\Lambda = 0$  for the time being, together with  $c = 0$ . We then get

$$\frac{d\phi}{da} = \pm \frac{\sqrt{6-n}}{\sqrt{a^2 \kappa^2 + \frac{n\varpi}{6f a^{n-2}} + \frac{f \tilde{\kappa}^4}{2n a^{4-n}}}}. \quad (36)$$

The relation  $V = V_0 e^{-k\phi}$  will only hold if  $\phi \propto \log a$ , so the question arises of whether that condition is compatible with Eq. (33). In order to find the answer we power expand  $d\phi(a)/da$  around  $a = \infty$  and keep only the first term in the series. It can be noticed immediately that the required condition  $\phi \propto \log a$  will only be satisfied asymptotically for large  $a$  if  $0 < n < 6$ . Therefore, the study will be restricted to just those cases. This would then give

$$V \sim f e^{-\sqrt{6-n} \kappa \phi}, \quad (37)$$

with

$$\phi \sim \frac{\sqrt{6-n}}{\kappa} \log a. \quad (38)$$

There are three exactly integrable cases, namely  $n = 2, 3, 4$ , and the corresponding potentials turn out to be functions of  $e^{-\kappa \sqrt{6-n} \phi}$ . For other values of  $n$  one can only give approximated expressions for late time limits. The leading terms in the solution  $\phi(a)$  will be the  $\varpi$ -dependent or the  $\tilde{\kappa}$ -dependent ones depending on whether  $0 < n < 3$  or  $3 < n < 6$ .

With the aim of giving the expressions in a form as simple as possible we define the coefficients

$$s_n = \frac{n \varpi}{24 f \kappa^2} \quad (39)$$

and

$$b_n = \frac{f \tilde{\kappa}^4}{8 n \kappa^2} = \frac{3f}{4 n \lambda}, \quad (40)$$

where the letters  $s$  and  $b$  indicate that they are related to the shear and the brane tension respectively. We outline the expressions for the scalar field and potential in Table II.

	$\phi(a)$	$V(\phi)$
$n = 2$	$\frac{1}{\kappa} \log \frac{a^2 + 2s_2 + \sqrt{a^2(a^2 + 4s_2) + 4b_2}}{2}$	$\frac{e^{2\kappa\phi} f}{((e^{\kappa\phi} - s_2)^2 - b_2)^2}$
$n = 3$	$\frac{2}{\sqrt{3}\kappa} \log \frac{a^{3/2} + \sqrt{a^3 + 4(b_3 + s_3)}}{2}$	$\frac{e^{\sqrt{3}\kappa\phi} f}{(e^{\sqrt{3}\kappa\phi} - b_3 - s_3)^2}$
$n = 4$	$\frac{1}{\sqrt{2}\kappa} \log \frac{a^2 + 2b_4 + \sqrt{a^2(a^2 + 4b_4) + 4s_4}}{2}$	$\frac{e^{\sqrt{2}\kappa\phi} f}{(e^{\sqrt{2}\kappa\phi} - b_4)^2 - s_4}$
$0 < n < 3$	$\frac{2\sqrt{6-n}}{n\kappa} \log \frac{a^{n/2} + \sqrt{a^n + 4s_n}}{2}$	$\frac{f e^{\sqrt{6-n}\kappa\phi}}{(e^{n\kappa\phi/\sqrt{6-n}} - s_n)^{2(6-n)/n}}$
$3 < n < 6$	$\frac{2}{\sqrt{6-n}\kappa} \log \frac{a^{3-n/2} + \sqrt{a^{6-n} + 4b_n}}{2}$	$\frac{e^{\sqrt{6-n}\kappa\phi} f}{(e^{\sqrt{6-n}\kappa\phi} - b_n)^2}$

TABLE I: Scalar fields and potentials for the integrable cases. The expressions in the last two rows are only valid in the large  $a$  approximation.

We have chosen the integration constant  $\phi_0$  in such a way that in the large  $a$  and large  $\phi$  limits we consistently recover the same expressions as in Eqs. (37) and (38). Solutions similar to these were obtained by Barrow and Saich [28] in a study of Friedmann universes containing a perfect fluid and a massive scalar field under the requirement that the kinetic and potential energies of the scalar field be proportional.

Before we go further, a remark regarding the choice of the value of  $\Lambda$  is in order. We have found three exact solutions for the  $\Lambda = 0$  case (see Table 1), but we can, by the same token, find another three for any non null value of  $\Lambda$ . This requires a little generalization that lies in including in the potential a term acting like a cosmological constant. To this end we set  $F(a) = fa^n + ha^6$  which will obviously yield  $V(a) = fa^{n-6} + h$ . Now, for the choice

$$h = -2 \frac{3\kappa^2 \pm \sqrt{9\kappa^4 - 3\Lambda\tilde{\kappa}^4}}{\tilde{\kappa}^4} \quad (41)$$

one gets

$$\frac{d\phi}{da} = \pm \frac{\sqrt{6-n}}{\sqrt{a^2(\kappa^2 + h\tilde{\kappa}^4) + \frac{n\varpi}{6fa^{n-2}} + \frac{f\tilde{\kappa}^4}{2na^{4-n}}}}. \quad (42)$$

This can be obtained from Eq. (33) upon the replacement  $\kappa^2 \rightarrow \kappa^2 + h\tilde{\kappa}^4$  and the same applies for the expression giving  $dt/da$ . Therefore, the results for the previous set of solutions can be easily extended to any  $\Lambda \neq 0$  value.

#### IV. KINEMATICS AND INFLATIONARY BEHAVIOR

The subject in hands now is the kinematics of our cosmological models. Let us first see if the energetic features of our cosmologies are suitable for inflation. A necessary condition for the occurrence of accelerated expansion is that the potential part of the energy density dominates, and for this to be possible the potential will have to be flat enough. Let us denote

$$\rho_{kin} = \frac{1}{2}\dot{\phi}^2 \quad (43)$$

and

$$\rho_{pot} = V, \quad (44)$$

so that  $\rho = \rho_{kin} + \rho_{pot}$ . Specifically, we have

$$\rho_{kin} = \frac{c}{a^6} + \frac{6-n}{n} \rho_{pot}. \quad (45)$$

This means that the kinetic and potential energy decrease at the same pace at late times (large  $a$ ) if  $c \neq 0$ , and at any time if  $c = 0$ . Moreover, if the latter holds we will have a barotropic equation of state  $p = (1 - n/3)\rho$ . This means we have found the exact potentials for radiation ( $n = 2, p = \rho/3$ ), dust ( $n = 3, p = 0$ ), and cosmic strings ( $n = 4, p = -\rho/3$ ) models. Note that, in the  $c \neq 0$  case the energy density could be dominated by its kinetic part at early times.

It has to be pointed out that there is one more integrable case of Eq. (33), the stiff fluid model ( $f = 0$ ), which was found in Ref. [22].

Typically, the inflationary behavior depends on the slope of the potential, but comparisons between potentials with different slopes must be done for identical values of the scalar field. In our models we have  $\partial^2 V / (\partial \varpi \partial \phi) \geq 0$  and  $\partial^2 V / (\partial \tilde{\kappa} \partial \phi) \geq 0$ . This can be checked using  $\partial V / \partial \phi = (dV/da)(da/d\phi)$  and Eqs. (35) and (33). This would seem to indicate that the anisotropy and the quadratic corrections would make inflation less likely.

Nevertheless, the definitive answer is provided by the deceleration factor  $q$ , which is defined through

$$q = -\frac{\ddot{a}a}{\dot{a}^2}, \quad (46)$$

and can be evaluated on calculating  $t = t(a)$  from Eq. (32). In Table 2 we summarize the behavior of  $q$  at late and early times.

		$t - t_0$	$q$
$a \rightarrow 0$	$0 < n < 3$	$\frac{na^{6-n}}{f(6-n)\tilde{\kappa}^2}$	$5 - n$
	$n = 3$	$\frac{a^3}{\sqrt{3\varpi + f^2\tilde{\kappa}^4}}$	2
	$3 < n < 6$	$\frac{a^3}{\sqrt{3\varpi}}$	2
$a \rightarrow \infty$	$0 < n < 6$	$\frac{\sqrt{2n} a^{3-n/2}}{(6-n)\sqrt{f}\tilde{\kappa}}$	$2 - \frac{n}{2}$

TABLE II: Asymptotic expressions for  $t$  as a function of  $a$ , together with the corresponding deceleration factors.

For large enough  $a$  we get  $q = (4 - n)/2$ , so inflation will only occur at late times if and only if  $n > 4$ . This result is completely equivalent to that obtained in [9] in the framework of general relativity. There is a persistent non inflationary behavior in the asymptotic regime of the models if the potential is too steep. For  $a \rightarrow 0$ , however, inflation will by no means occur.

Let us push a bit further the fact that the sign changes in  $q$  mark the transition between an inflationary and a non inflationary epoch. As follows from Eq. (46) accelerated expansion will occur if  $\ddot{a} > 0$ , which in our models holds if

$$3(n-4)na^{6+n}f\kappa^2 + 3(n-5)a^{2n}f^2\tilde{\kappa}^4 - 2n^2\varpi a^6 > 0. \quad (47)$$

We see that  $n > 4$  is a necessary, but not sufficient, condition for accelerated expansion, and also a sufficient condition for  $a \rightarrow \infty$ , as proved above. Nevertheless, Eq.

(47) has in store some interesting information that will highlight how peculiar inflation is in brane cosmology.

Let us make  $a_{\text{inf}}$  stand for the value of the expansion rate above which the model expands. We will now study, by numerical techniques, how  $a_{\text{inf}}$  depends on the parameters of the model. First, we will focus on the dependence on the equation of state. After that, we will turn to consider the effect of the magnitude of the anisotropy and the brane tension.

Numerical calculations indicate that, in the relativistic limit, as is well known, the larger  $n$ , the smaller  $a_{\text{inf}}$  (i.e. the earlier inflation begins), whatever the value of  $\varpi$ . Inspection suggests, however, that this monotonic behaviour may not hold for the brane case, because the term in Eq. (47) associated with the extra dimensional corrections changes sign at  $n = 5$ . Moreover, the values of  $a_{\text{inf}}$  in the vicinity of  $n = 5$  must be very much alike the values corresponding to the relativistic limit (because the extra dimensional corrections will be negligible). Numeric evaluations show that for  $n$  close to 4 the behaviour of  $a_{\text{inf}}$  is almost identical in the brane and the relativistic case. However, as  $n = 5$  is approached, the monotonic decrease ceases and  $a_{\text{inf}}$  begins to grow with  $n$ , till it reaches a maximum value before  $n = 5$ . Note that the value of the maximum of  $a_{\text{inf}}$  does not depend on  $\varpi$ . From that point on, there is again monotonic decrease, but is more pronounced than in the relativistic case. Summarizing, the conditions for inflation are rather different in the brane and relativistic setups. Nevertheless, several restrictions on the parameters of the models have to be considered. On the one hand the potential must remain much smaller than the fourth power of the four-dimensional Planck's mass  $M_P$  so that classical physics is valid. On the other hand, the five-dimensional Planck's mass  $\tilde{M}_P$  is typically much smaller than its four-dimensional counterpart. In Fig. 1 we have represented  $\log a_{\text{inf}}$  as a function of  $n$ , for several values of  $\varpi$ , within the ranges  $4 < n < 5$  and  $1/2 \leq \varpi \leq 8$  respectively, choosing  $\tilde{M}_P$  considerably smaller than  $M_P$ .

We have seen, also by numeric means, that  $a_{\text{inf}}$  increases with the shear and decreases with the brane tension<sup>1</sup>. However, those very changes in the parameters which act to delay inflation, make the Hubble factor grow in turn [29], and inflation can be sustained by potentials that would not be able to do so in a relativistic setup [30]. Thus, both the anisotropies and the extra dimensional corrections seem to act locally in a way conducive to inflation. In order to determine whether this is also the case in a global way, it would be necessary to evaluate the number of e-foldings between the beginning and the end of inflation. So, we would have to find physical arguments allowing to know when inflation ends, and then calculate the amount of inflation achieved while acceler-

<sup>1</sup> The shear increases with  $\varpi$ , but the brane tension decreases as  $\tilde{\kappa}$  increases.

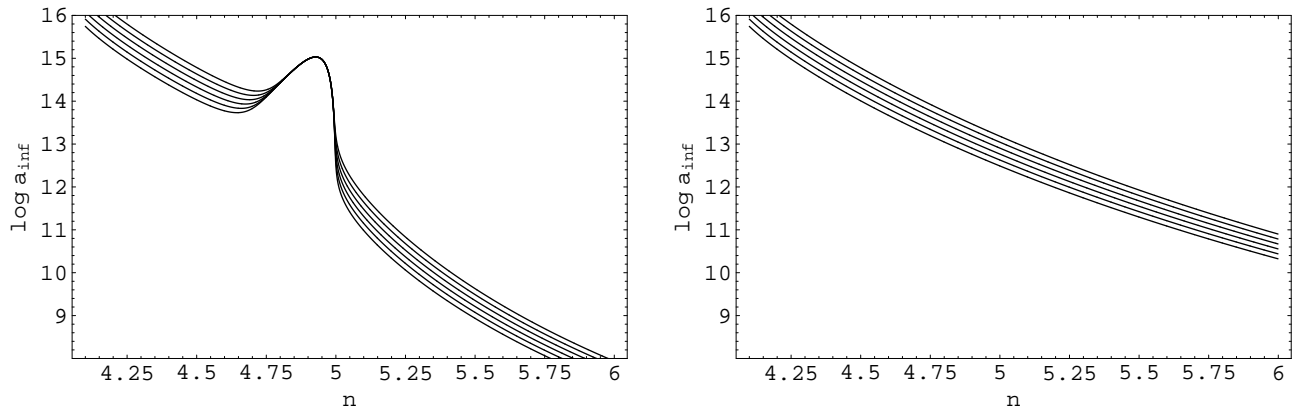


FIG. 1: Plot of  $\log a_{\text{inf}}$  as a function of  $n$  for  $\varpi = 0.25, 0.5, 1, 2, 4, 8$ ,  $a(0)=1$ ,  $f = 10^{-30}$ ,  $\kappa = 8\pi$ , and  $\tilde{\kappa} = 8\pi \times 10^9$ . for the figure on the left and  $\tilde{\kappa} = 0$  for the figure on the right. Here the Planck mass on the brane  $M_P$  has been set to one. Lower curves in each plot correspond to lower values of  $\varpi$ .

ated expansion persists.

Unlike what happens with the time at which inflation ends, we can determine for our models the time at which inflation begins. We have seen that the larger  $\varpi$  or  $\tilde{\kappa}$ , the larger  $a_{\text{inf}}$ . But because of the way the Hubble factor behaves, the value of  $\Delta t$  associated with a given value of  $a$  grows too. Due to the lack of exact expressions for the  $t(a)$  curves for the inflationary cases ( $n > 4$ ) one can only find the corresponding value of time by using approximate solutions or numerical integration; we will choose the second method. In Fig. 2 we have plotted the  $t(a)$  curves for  $n = 4.5$  and for the same values of  $\tilde{M}_P$  and  $\varpi$  as in Fig. 1; we have also indicated in the picture the value of  $a_{\text{inf}}$  as a function of time.

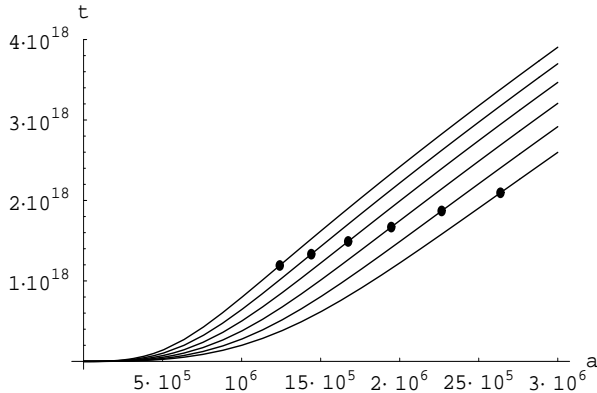


FIG. 2: Plot of the  $t(a)$  curves for  $\varpi = 0.25, 0.5, 1, 2, 4, 8$ ,  $a(0)=1$ ,  $f = 10^{-30}$ ,  $n = 4.5$ ,  $f = 10^{-30}$ ,  $\kappa = 8\pi$  and  $\tilde{\kappa} = 8\pi \times 10^9$ . Here the Planck mass on the brane  $M_P$  has been set to one. Lower curves correspond to higher values of  $\varpi$ . Each dot marks the beginning of inflation

For the sake of completeness and comparison, let us investigate whether our models isotropize or not. We have seen above that the influence of the anisotropies in the inflationary behavior becomes less and less important

as time goes by. The anisotropic shear parameter, which is defined as

$$\Omega_{\text{shear}} = \frac{\sigma^2}{6H^2} = \frac{\sum_{i=1}^3 (H_i - H)^2}{6H^2}, \quad (48)$$

reflects that too. Using Eqs. (21)-(23), and after a little algebra, we get

$$\Omega_{\text{shear}} = \frac{\varpi P^2}{3H^2}, \quad (49)$$

and according to Eq. (33) one finally has

$$\Omega_{\text{shear}} = \frac{\varpi a^6 n^2}{6(2a^{6+n} f \kappa^2 + 3a^{2n} f^2 \tilde{\kappa}^4 + a^6 n^2 \varpi)}. \quad (50)$$

In the inflationary cases, since  $n > 4$ , we see that  $\Omega_{\text{shear}}$  decreases monotonically with time, tending to a null value. Moreover, the larger  $n$  and  $\tilde{\kappa}$  the more rapidly it decreases. Thus, although we have genuinely anisotropic models at early times, anisotropy gets completely dissipated in the course of the evolution. So, our models do isotropize at late times, and both faster inflation and extra dimensional effects make the isotropization process more efficient.

## V. GENERALIZATION TO MULTI-FIELD MODELS

Let us now take advantage of the results in the previous sections to discuss how the situation would change if we had multiple fields. Assisted inflation [31], in its simplest form, is realized with configurations of identical multiple fields with no interaction among themselves, but which self-interact through the same potential (see also [33, 34]). Aspects of this proposal in connection with the brane scenario have been discussed in [32, 35, 36].

If we now make  $\phi$  denote each scalar field and  $V(\phi)$  each individual potential, we can obtain the multifield version of our models just by making the replacement

$$\kappa \rightarrow \sqrt{N}\kappa, \quad (51)$$

$$\tilde{\kappa} \rightarrow \sqrt{N}\tilde{\kappa}, \quad (52)$$

where  $N$  denotes the number of fields. It can be immediately seen than the larger  $N$ , the larger the Hubble factor, so that the assisted inflation proposal is valid in the brane scenario too.

The asymptotic individual potentials will become

$$V \sim e^{-\kappa\sqrt{N(6-n)}\phi}. \quad (53)$$

Thus, an increase in the number of fields makes the potentials turn flatter, which means that late time inflation becomes likelier.

The condition for having  $q < 0$  reads now

$$3(n-4)na^{6+n}f\kappa^2N + 3(n-5)a^{2n}f^2\tilde{\kappa}^4N^2 - 2n^2\varpi a^6 > 0, \quad (54)$$

so, the larger  $N$ , the less anisotropy influences the occurrence of inflation. This is also reflected by the fact that  $\Omega_{\text{shear}}$  decreases with growing  $N$ . Nevertheless  $n > 4$  is still necessary for inflation, no matter how large  $N$  is.

The behaviour of  $a_{\text{inf}}$  as a function of  $n$  will change when  $N$  grows. We have seen in the previous section that, unlike in the relativistic case,  $a_{\text{inf}}$  is not a monotonically decreasing function of  $n$ , and that it has a maximum near  $n = 5$ . It is not difficult to see that a larger  $N$  increases the value of  $a_{\text{inf}}$  at the maximum and makes wider the range of  $n$  in which  $a_{\text{inf}}$  grows with growing  $n$ . Summarizing, the addition of fields accentuates the differences between relativistic and brane cosmologies.

## VI. CONCLUSIONS AND FUTURE PROSPECTS

We have studied Bianchi I type brane cosmologies with barotropic perfect fluids, and we have shown that they can be interpreted in the language of scalar fields as exponential potential models. From the expressions here obtained, it follows that the models behave, at late times, as if the potential had only one exponential term. Consistently, the differences between the brane and relativistic models get washed off as the models evolve in time. The effects of the anisotropy disappear in a similar fashion as well.

Our main focus point has been the average evolution of the models, in comparison to their relativistic counterparts. In particular, we have studied the features of the inflationary behaviour that can be traced back to extra dimensional effects but not to the bulk.

It is a well-known fact that, in the relativistic context, the condition  $3p < -\rho$  (which corresponds to  $n > 4$ ) is necessary and sufficient for inflation to occur. Expressed in terms of the potential, it translates into the requirement that the slope of the asymptotically dominating potential be smaller than  $\sqrt{2}\kappa$ . In brane cosmology, that condition remains necessary for inflation, although we show that it is not sufficient.

Using numeric means we have found some other new interesting results in connection with the kinematics of the models. In the relativistic case, the value of the average expansion factor at the beginning of inflation decreases monotonically as the ratio  $p/\rho$  becomes more negative. In the brane models, the behaviour is basically the same, except for the vicinity of  $3p < -2\rho$ , where the monotonic decrease turns into monotonic increase, until a local maximum in the function is reached. Moreover, the value of the expansion rate at that extreme point does not depend on the amount of anisotropy.

Taking into account the tight restrictions to the parameters of the models set by physical arguments, our calculations show that the effect of the anisotropy in the evolution will only be noticeable if inflation begins very late, which is not a desirable situation.

We have discussed a possible realization of the assisted inflation proposal, which is underrepresented in the literature on brane cosmology. In particular, we have considered generalizations to multi-field configurations of non-interacting scalar fields. The conclusion reached is that the aforementioned requirement on the slope of the potential still applies. On the other hand, an increase in the number of fields  $N$  will make the Hubble factor grow, so that inflation will be likelier. This means that the brane version of the assisted inflation proposal is also feasible. Moreover, as  $N$  grows the models become more isotropic.

A possible extension to this work would be exploring the possibility of generalizing these solutions to include bulk effects. It would also be interesting to carry out similar studies for anisotropic spacetimes with non-trivial space-curvature. Bianchi V type models would be the obvious candidate for a first attempt because they are the simplest generalization of the open FRW models.

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